Hearts, Scissors, and Clovers, Oh My!

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Abstract

Common Euclidean polar coordinate graphs are explored in taxicab geometry resulting, as usual, in a batch of pleasant surprises.

In Euclidean geometry some of the most interesting graphs that use very simple expressions occur in polar coordinates. Limaçons, lemniscates, roses, and the spiral of Archimedes all have simple representations in polar coordinates given the variety and beauty of their graphs.

At the heart of polar coordinates are the two variables r, the distance from the origin, and the angle θ, the direction in which to travel the distance r. Because taxicab geometry employs a different distance function, the expression for and the meaning of r have changed. Furthermore, because angles are also a type of distance measure, angles and trigonometric functions have changed as well. All of these leave open the possibility for many twists and turns in the road on a tour of polar graphs in a taxicab.

It should be noted that this research is being conducted in pure taxicab geometry using native taxicab angles (see [1]) and not in modified taxicab geometry in which Euclidean angles are retained.

1 Taxicab Geometry

Taxicab geometry can be thought of as a simple adjustment to Euclidean geometry: a change to the distance function (metric) for the space. As illustrated in works like [2], the taxicab metric can be thought of as measuring distance in the way a taxicab would when driving a perfect city with horizontal and vertical streets: $d = |x| + |y|$. From the post office, the theater may be closer (six units) than the courthouse (seven units) for a taxicab, but the opposite is true for the proverbial crow (see Figure 1).

The concept of distance has far-reaching effects in a geometry. Everything from angles [1] to conic sections [3] to area and volume [4] is in some manner dependent on the meaning of distance. For other interesting research on taxicab geometry see [5, 6, 7, 8, 9].

2 Taxicab Angles and Trigonometry

As mentioned above, the definition of native angles in a geometry involves the metric present on the space. When the metric is changed the shape of a circle

is changed and therefore the measurement of arc length along the unit circle to define angle measure has changed [1]. When a native taxicab angle measure called a t-radian is defined even the value of π changes [1, 10] to an integer of $\pi_t = 4$! For our purposes, the single greatest impact is the shape of a circle, which now resembles a diamond (Figure 2).

When native trigonometric functions are defined in taxicab geometry, the sine and cosine functions become discontinuous piecewise linear functions [1]. However, in our investigations below, we can avoid the messiness of dealing with the piecewise cosine and sine definitions by noting that the mnemonic SOH-CAH-TOA still holds for "properly oriented" taxicab right triangles. A right triangle is "properly oriented" if the sides of the triangle forming the right angle are parallel to the coordinate axes as shown in Figure 3. With this orientation we show below that the trigonometric functions of an angle can be computed using the mnemonic.

The size of $\angle B$ in Figure 3 can be determined using the Arc Length Theorem (Theorem 2.6 citeThompson-Trig). A taxicab circle of radius c about point B (gray) subtend an arc of length $s = c \cdot m(\angle B)$. In this case, $s = 2b$ and therefore, $m(\angle B) = \frac{2b}{c}$. Using the definition of cosine as $\cos\theta = 1 - \frac{1}{2}\theta$, we have:

$$\cos B = 1 - \frac{1}{2}m(\angle B) = 1 - \frac{1}{2}\frac{2b}{c} = \frac{c-b}{c} = \frac{a}{c} = \frac{\text{adjacent}}{\text{hypotenuse}} \tag{1}$$

since $c = a + b$. Similarly, using $\sin\theta = \frac{1}{2}\theta$

$$\sin B = \frac{1}{2}m(\angle B) = \frac{1}{2}\frac{2b}{c} = \frac{b}{c} = \frac{\text{opposite}}{\text{hypotenuse}} \tag{2}$$

With these tools, we will be able to replace sine and cosine functions in various polar equations with Cartesian equivalents in the same manner in both Euclidean and taxicab geometries.

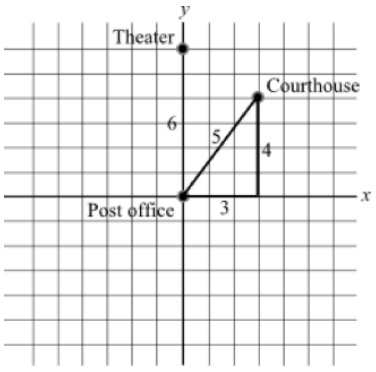

Figure 1: A city-dweller's view of shortest distance is different than a crow's: from the post office, the theater is closer than the courthouse for a taxicab.

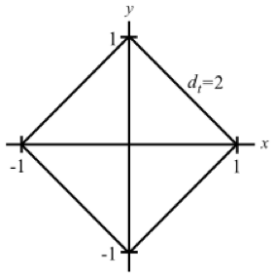

Figure 2: A unit circle in taxicab geometry.

3 Taxicab Polar Graphs

We will investigate these graphs using their Cartesian coordinate equivalents, partly to highlight how the distance function impacts the expressions but also because a taxicab polar coordinate graphing calculator does not exist!

3.1 Limaçons

Limaçons are generated using expressions such as $r = a \pm b\sin\theta$ and $r = a \pm b\cos\theta$. If we select the positive sine version and express it in Cartesian coordinates we have

$$\sqrt{x^2+y^2} = a + b\frac{y}{\sqrt{x^2+y^2}} \tag{3}$$

where we have used $\sin\theta = \frac{y}{r}$ on the right-hand side. If we further rearrange to avoid division by zero and include the positive and negative square roots the result is

$$x^2 + y^2 = \pm a\sqrt{(x^2+y^2)} + by \tag{4}$$

We can see that the Euclidean distance function appears twice here, once explicitly for r itself and again implicitly for r as the denominator of the sine

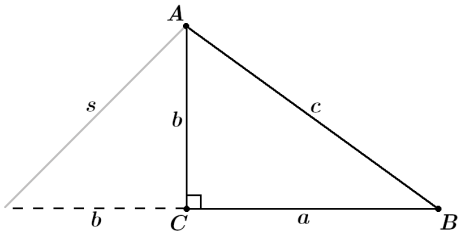

Figure 3: A "properly oriented" right triangle in taxicab geometry, with a partial taxicab circle of radius c about point B (gray), showing the arc subtended by $\angle B$ on the circle

function. If we replace these with the taxicab distance formula we arrive at our destination:

$$(|x| + |y|)^2 = \pm a\,(|x| + |y|) + b \tag{5}$$

Various taxicab limaçon graphs are shown with their Euclidean counterparts in Figure 4 using values of a/b that produce markedly different shapes. Note that the ranges of a/b for loops ($a/b < 1$), dimples ($1 < a/b < 2$), and convexity $a/b \geq 2$ appear to have the same boundaries as in Euclidean geometry.

A limaçon with $a = b$ is called a cardioid (from Greek "kardia" meaning heart) because it has the general shape of a heart. In this case, taxicab geometry wins the prize because its cardioid is shaped more like a heart than a cardioid in Euclidean geometry!

3.2 Lemniscates

A lemniscate is a special case of a Cassini oval, which was studied extensively in taxicab geometry from a different perspective in [11]. Polar equations such as $r^2 = \pm a^2 \cos(2\theta)$ and $r^2 = \pm a^2 \sin(2\theta)$ produce lemniscates that have the shape of a figure '8' or an infinity symbol. For $r^2 = a^2 \cos(2\theta)$, a double-angle

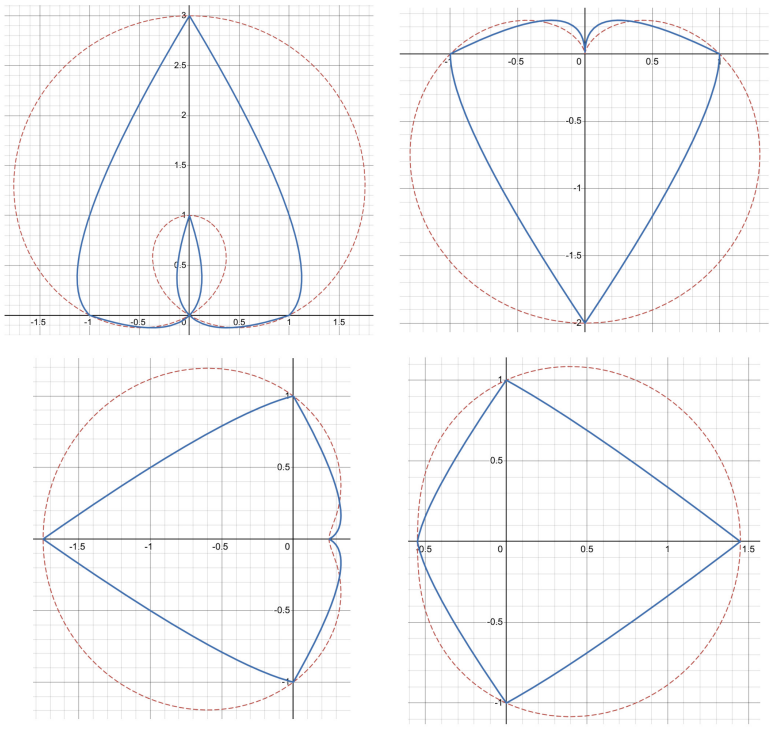

Figure 4: Limaçons in Euclidean (dashed) and taxicab (solid) geometries

identity, substitution, and algebra produces the equivalent Euclidean Cartesian equation:

$$\left(x^2+y^2\right)^2 + a\left(x^2+y^2\right) = 2ax^2 \tag{6}$$

along with the taxicab Cartesian equation

$$\left(|x|+|y|\right)^4 + a\left(|x|+|y|\right)^2 = 2ax^2 \tag{7}$$

Graphs of a Euclidean and taxicab lemniscate are shown in Figure 5.

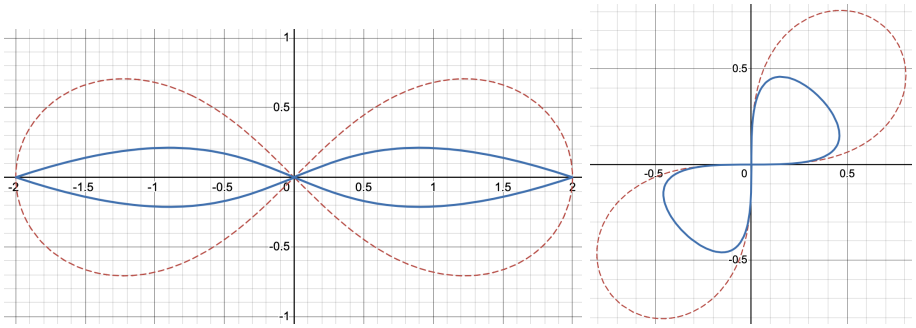

Figure 5: Lemniscates in Euclidean (dashed) and taxicab (solid) geometries

3.3 Roses

Without a taxicab polar coordinate graphing utility, roses will be more difficult to investigate. However, a few reasonable examples can be produced.

A rose is produced from polar equations of the form $r = a\sin(n\theta)$ or $r = a\cos(n\theta)$. If n is odd, the rose has n petals, whereas if n is even, there are $2n$ petals. Double-angle trigonometric identities can be employed for $n = 2$ to easily produce Cartesian versions of the polar equations. For instance, the 4-petal rose has the Euclidean form

$$r = a\sin(2\theta) = 2a\sin\theta\cos\theta \tag{8}$$

which yields the Euclidean Cartesian equation

$$\pm(x^2+y^2)^{3/2} = 2axy \tag{9}$$

where we have again introduced the plus and minus owing to the even root. The equation for the corresponding taxicab rose with four petals is

$$\pm(|x|+|y|)^3 = 2axy \tag{10}$$

When graphed, the utility just happened to choose green for the color of the taxicab curve which further enhanced the impression that these were not roses but clovers (Figure 6)!

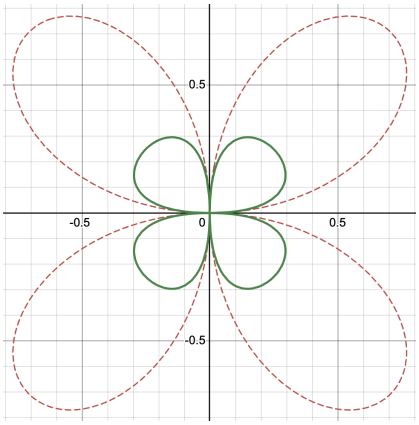

Figure 6: The Euclidean 4-petal rose (dashed) becomes a 4-leaf clover (solid) in taxicab geometry

For the case where n is odd, consider $r = a\cos(3\theta)$. After some work with additive and double-angle trigonometric identities, the equivalent Cartesian equation is

$$(x^2 + y^2)^2 = a\left(2x^3 - x(x^2 + y^2) - 2xy^2\right) \tag{11}$$

with taxicab equivalent

$$(|x| + |y|)^4 = a\left(2x^3 - x(|x| + |y|)^2 - 2xy^2\right) \tag{12}$$

Figure 7 shows the result which apparently should be called "taxicab scissors." Note the loss of rotational symmetry in the taxicab scenario.

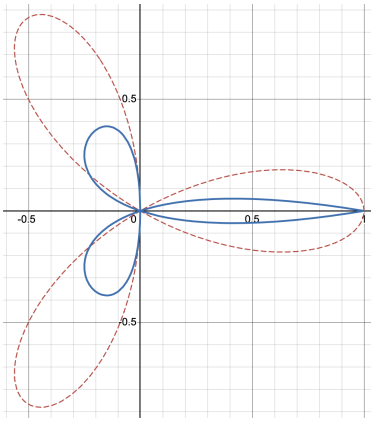

Figure 7: A Euclidean 3-petal rose (dashed) and corresponding taxicab scissors (solid)

3.4 The Spiral of Archimedes

The final basic polar graph we examine is probably the simplest of all: the Spiral of Archimedes which has the trivial polar equation of $r = \theta$. Therefore, we need a strategy to convert a taxicab angle θ measured in t-radians to Cartesian coordinates. In Quadrant I, an angle θ in standard position whose terminal ray intersects a circle of radius r at point (x, y) subtends an arc length of $2y$ along the line $y = -x + r$ of the circle. Again using Theorem 2.6 [1], the relationship $s = r\theta$ still holds which means that

$$\theta = \frac{s}{r} = \frac{2y}{r} = \frac{2y}{|x| + |y|} \tag{13}$$

So, the equation for the spiral in Quadrant I is

$$r = |x| + |y| = \frac{2y}{|x| + |y|} = \theta \tag{14}$$

In Quadrant II the measure of the angle is 2 t-radians plus an acute angle with initial side along the positive y-axis that subtends an arc of length $2|x|$. Because x is negative, this leads to

$$\theta = 2 - \frac{2x}{|x| + |y|} = 2\left(1 - \frac{x}{|x| + |y|}\right) \tag{15}$$

In general, for Quadrant n (with Quadrant V identical to Quadrant I, etc.)

$$\theta = \begin{cases} 2\left((n-1) + \frac{(-1)^{\frac{n-1}{2}} y}{|x|+|y|}\right), & n \text{ odd} \\ 2\left((n-1) + \frac{(-1)^{\frac{n}{2}} x}{|x|+|y|}\right), & n \text{ even} \end{cases} \tag{16}$$

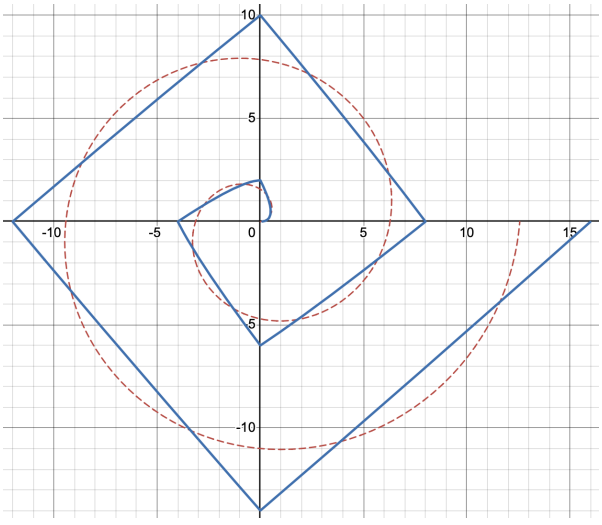

Figure 8: The Euclidean (dashed) and taxicab (solid) spiral of Archimedes

For example, this leads to the following equations for the spiral in Quadrants II and III:

$$|x| + |y| = 2\left(1 - \frac{x}{|x| + |y|}\right) \tag{17}$$

$$|x| + |y| = 2\left(2 - \frac{y}{|x| + |y|}\right) \tag{18}$$

The result for $[0, 4\pi]$ is shown in Figure 8.

None of the taxicab spiral sections are straight lines, but their curvature decreases with increasing r in the same manner as the Euclidean spiral, so that they quickly begin to have a linear appearance.

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